An introduction to Construction Schemes

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$\mathcal{F} \subseteq [\omega_1]^{<\omega}$ is a construction scheme of type $\{(m_{k+1}, n_{k+1}, r_{k+1})\}_{k\in\omega}$ if there is a decomposition $\mathcal{F} = \bigcup_{k\in\omega} \mathcal{F}_k$ such that for all $k \in \omega$:

1) $\mathcal{F}_0 = [\omega_1]^1$
2) $\mathcal{F}$ is cofinal in $[\omega_1]^{<\omega}$
3) If $F \in \mathcal{F}_k$, then $|F| = m_k$
4) If $F, E \in \mathcal{F}_k$, then $F \cap E \sqsubseteq F, E$

5) For all $F \in \mathcal{F}_{k+1}$, there are $\{F_i \mid i < n_{k+1}\} \subseteq \mathcal{F}_k$ and $R(F)$ such that:

a) $F = \bigcup_{i<n_{k+1}} F_i$

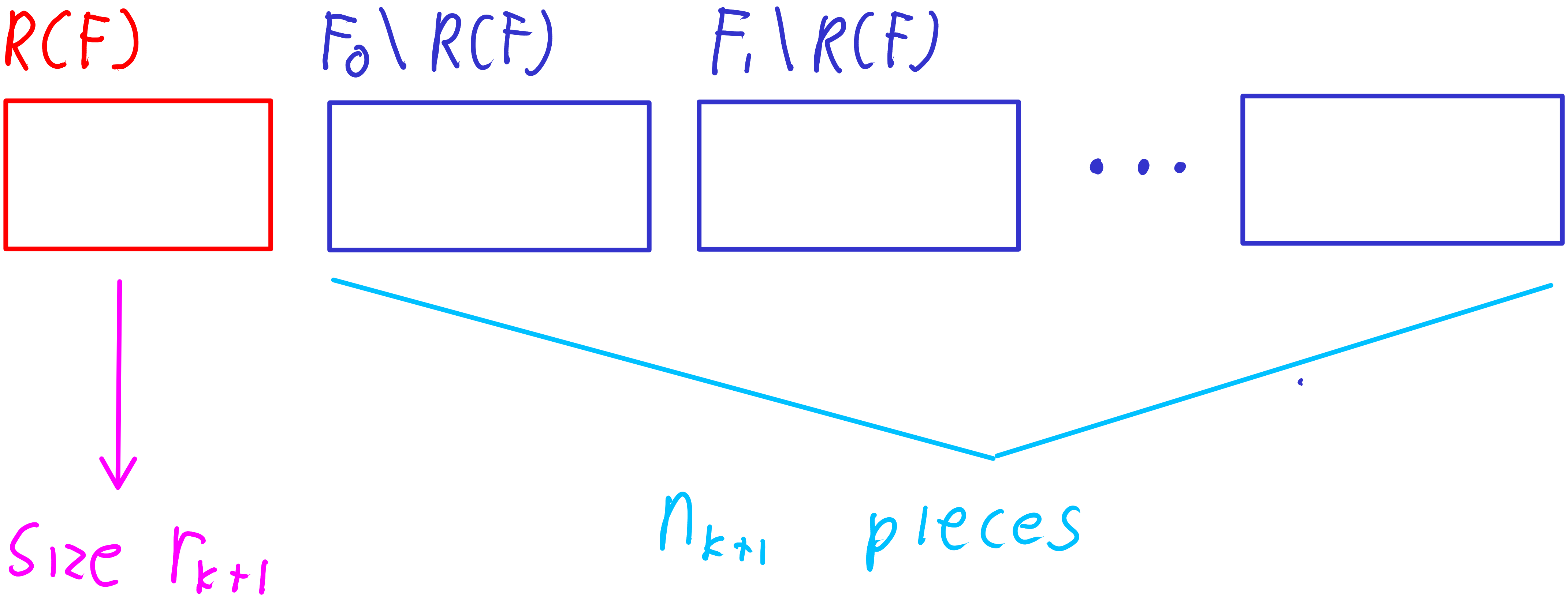
b) $\{F_i \mid i < n_{k+1}\}$ is a Δ-system with root $R(F)$

c) $|R(F)| = r_{k+1}$

d)

$$R(F) < F_0 \setminus R(F) < F_1 \setminus R(F) < \ldots < F_{n_{k+1}-1} \setminus R(F)$$

Each $F \in \mathcal{F}_{k+1}$ looks like this:

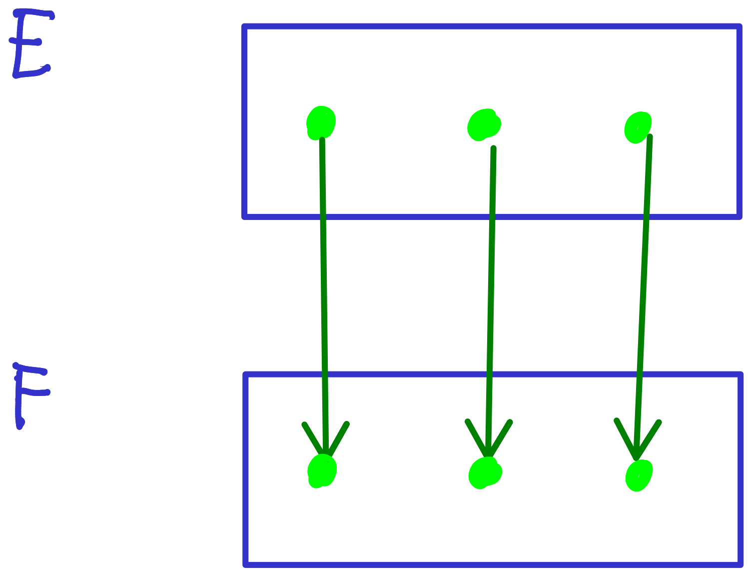


Theorem (Todorcevic)

Construction Schemes exist (for any type)

Def

Let $E, F \in [\omega]^{<\omega}$ of the same size. Denote by φ_{EF} the only increasing bijection from E to F.



As mentioned uncountably many times, Construction schemes simply exist (their existence follows from the axioms of ZFC).

We can demand stronger properties to the construction scheme.

Construction schemes with this properties can not be proved only from ZFC.

With this construction scheme with super powers, we can give alternative constructions of objects whose existence is independent from ZFC

This all powerful construction schemes are known as capturing schemes

Let $a_0, a_1, \ldots, a_l \in [\omega_1]^{<\omega}$ disjoint of the same size, $k \in \omega$ and $F \in \mathcal{F}_l$. We say that F captures $\{a_0, a_1, \ldots, a_l\}$ if:

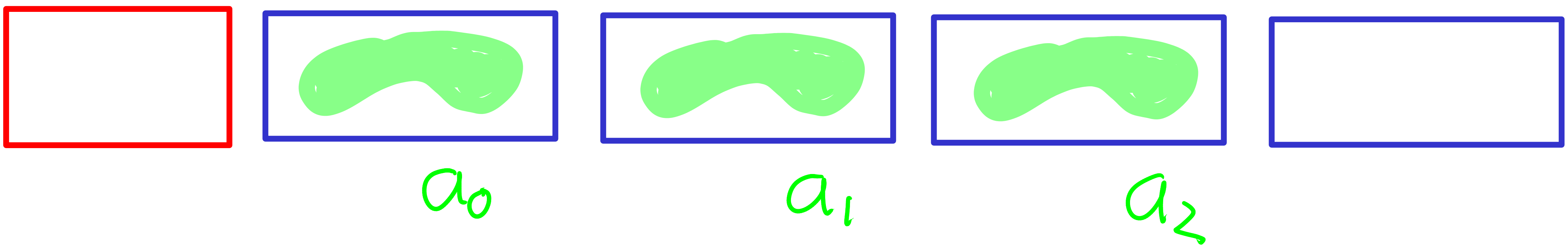
1) $\ell \leq n_k$

1) $l \leq n_k$

This means, F has at least l blocks (l is the number of sets we are capturing)

2) $a_i \subseteq F_i$ for all $i < \ell$

where $\{F_j \mid j < n_\ell\}$ is the canonical decomposition of F.



3) For all $i<\ell$ we have that

$$\varphi_{F_0 F_i}[a_0] = a_i$$

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This means that all of the a_i's are placed in the same way in the F_i's.

Example

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2) Let $n_k = 4$

3) Assume each $F_i \setminus R(F)$ have size 3

What does it mean that

F captures $\{a_0, a_1, a_2\}$?

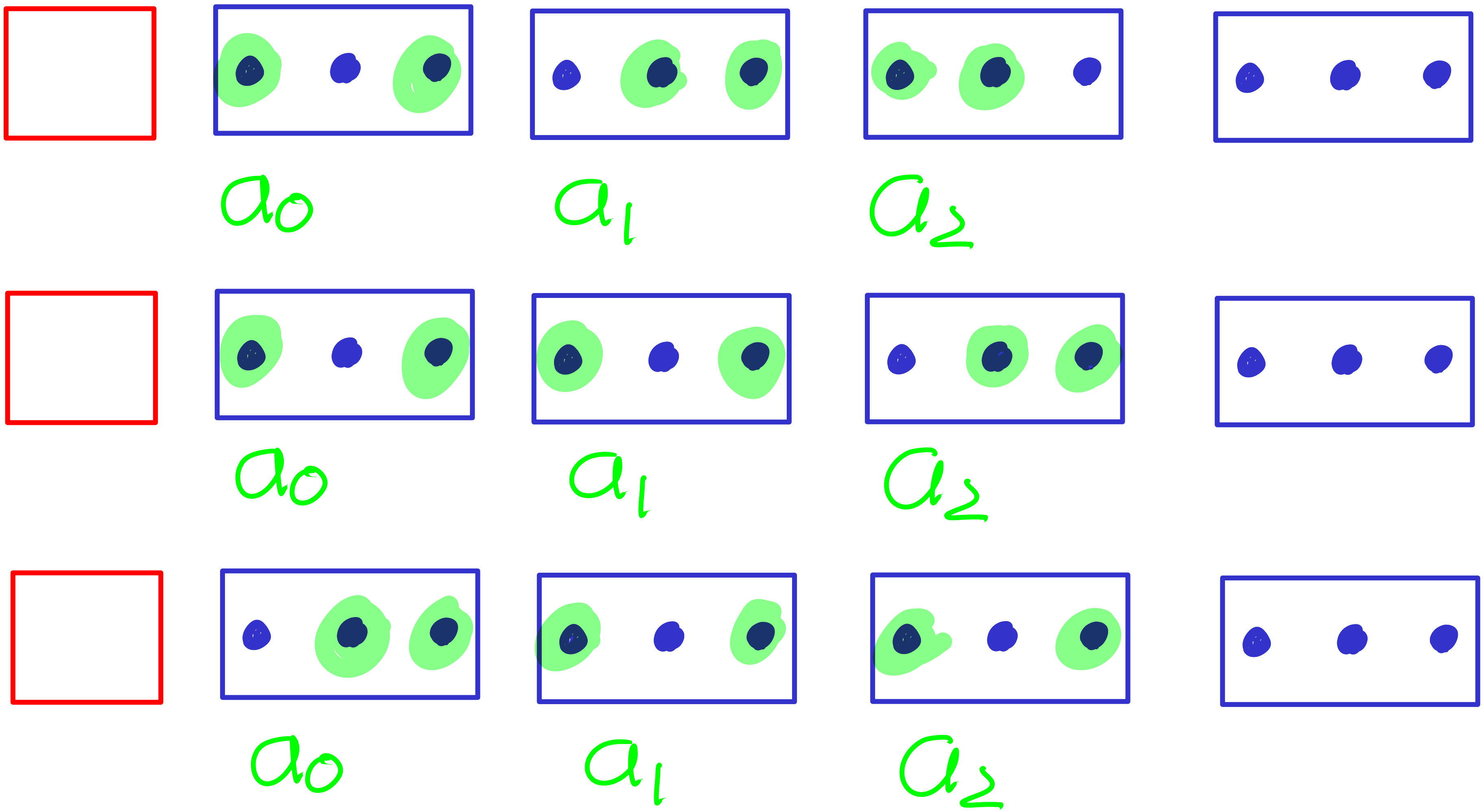
Here, F consists of 4 blocks. First we need that

$$a_0 \subseteq F_0 \setminus R(F)$$

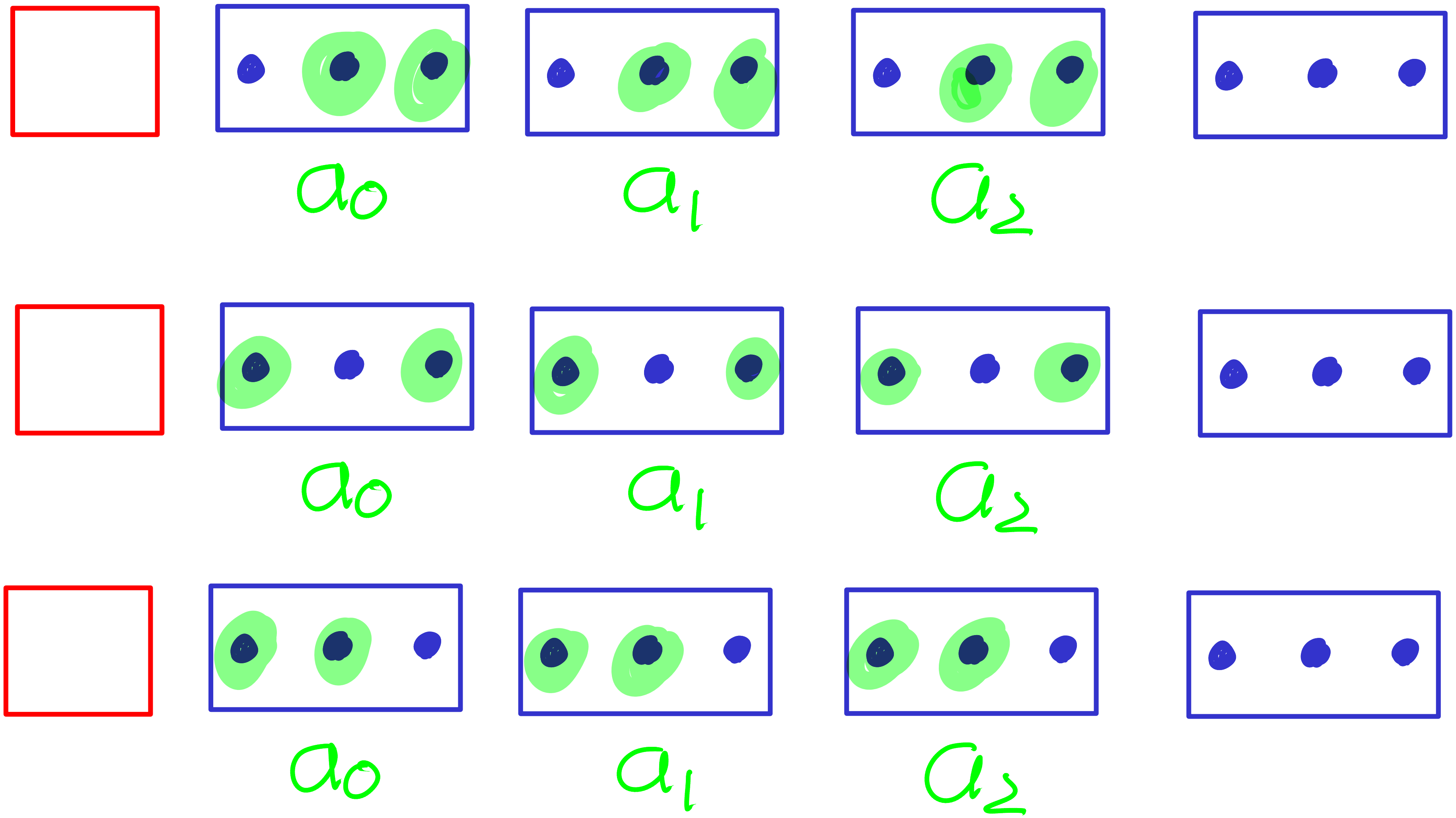
$$a_1 \subseteq F_1 \setminus R(F)$$

$$a_2 \subseteq F_2 \setminus R(F)$$

This are NOT capturing



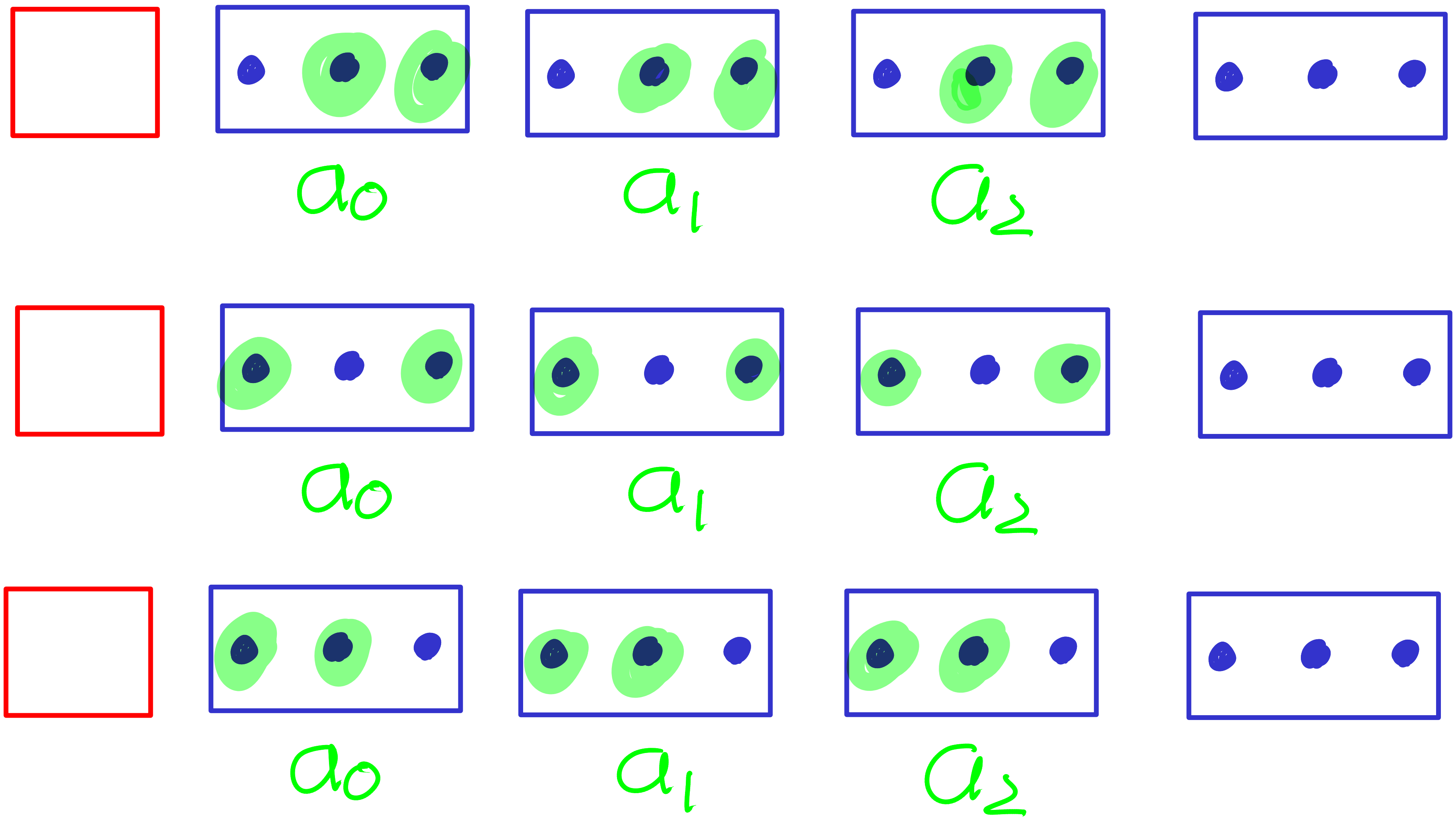
This are capturing



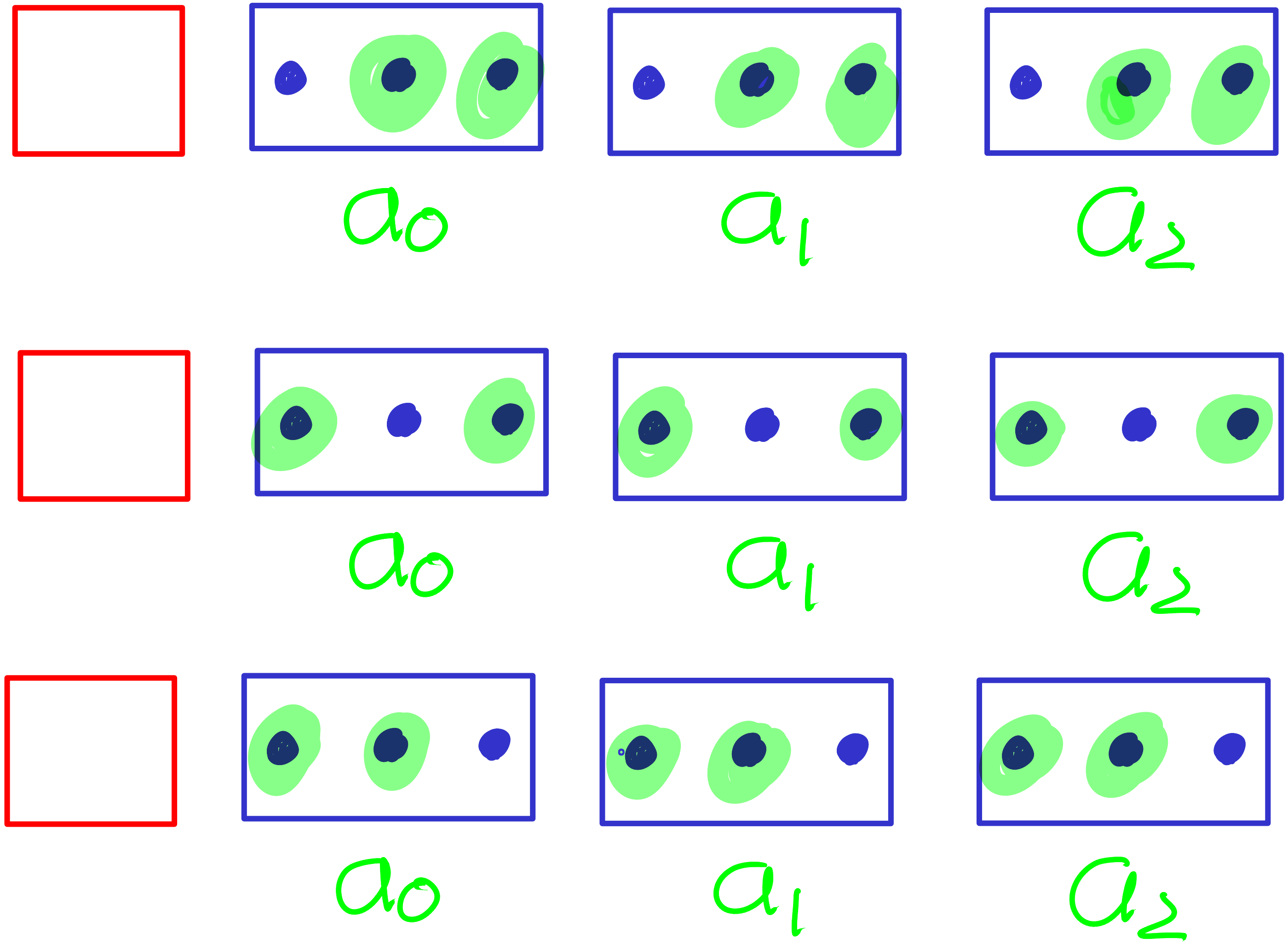
Moreover, we say that F fully captures $\{a_0, a_1, a_2\}$ if F captures $\{a_0, a_1, a_2\}$ and $n_k = \ell$

This means that there are no "free blocks" ie, every block contains an a_i.

NOT full capturing



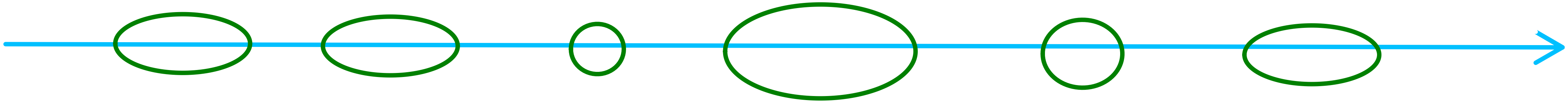
Full capturing

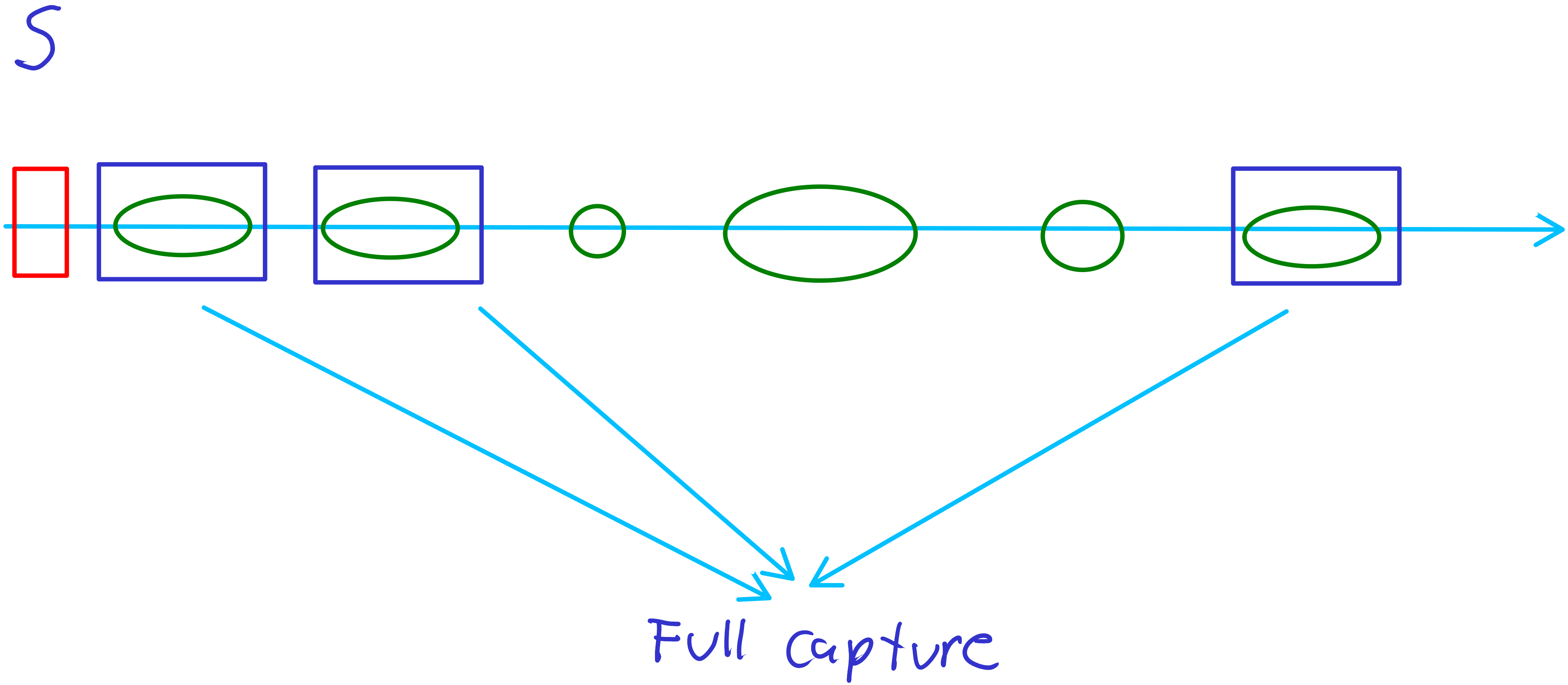


Def

We say $\mathcal{F}$ is fully capturing if for every uncountable $S \subseteq [\omega_1]^{<\omega}$ of disjoint sets and $k \in \omega$, there are $l > k$, $F \in \mathcal{F}_l$ and $\mathcal{C} \in [S]^{<\omega}$ such that F fully captures $\mathcal{C}$

S




S
Full capture

Sometimes we need a litte bit more:

Def

Let $\mathcal{P}$ be a partition of ω.

We say $\mathcal{F}$ is $\mathcal{P}$-fully capturing if for every uncountable $S \subseteq [\omega_1]^{<\omega}$ of disjoint sets, $P \in \mathcal{P}$ and $k \in \omega$, there are $l > k$, $l \in P$, $F \in \mathcal{F}_l$ and $e \in [S]^{<\omega}$ such that F fully captures e

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The point of the partition is that in different elements of the partition, we want to make different types of amalgamation

Theorem (Cruz-Chapital, G., Todorcevic)

Let $\alpha \in \omega \cup \{\omega\}$

$\diamondsuit \Rightarrow$ There is a partition $\mathcal{P}$ with $|\mathcal{P}| = \alpha$ and $\mathcal{F}$ a construction scheme (of any type) that is $\mathcal{P}$-fully capturing

A coherent Suslin tree

Def

A Suslin tree is a tree such that:

1) It has height ω_1

2) Its levels are countable

3) It has no cofinal branches

4) It has no uncountable antichains

Theorem (Jensen)

$\Diamond \Rightarrow$ There is a Suslin tree

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We do a proof using capturing schemes

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This means that the set

$$\{ \xi < \alpha \mid e_\alpha(\xi) \neq e_\beta(\xi) \}$$

is finite

If $\vec{e} = \langle e_\alpha \rangle_{\alpha < \omega_1}$ is a coherent sequence, we can define a tree:

$$T(\bar{e}) = \{ e_\alpha \mid \xi \leq \alpha < \omega_1 \}$$

This is a tree with countable
levels and height ω_1

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By choosing the sequence wisely, we can make sure the associated tree is Suslin

A Suslin tree T is coherent if there is a coherent sequence $\bar{e} = \langle e_\alpha \rangle_{\alpha < \omega_1}$ such that $T = T(\bar{e})$

Fix a type $\{(m_k, n_k, r_k)\}$ such that $n_k \geq 2^{(m_k - r_k)} + 1$ for all $k \in \omega$

$2^{(m_k - r_k)}$ is the size of the power set of

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We will use it to build a coherent Suslin tree

Here we will need to use to

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This is the reason we use a partition with 2 pieces

We start by defining a natural forcing for adding a coherent sequence with finite conditions

Define $\mathbb{P}$ as the set of all $p = (X, \xi)$ where:

1) $X \in [\omega_1]^{<\omega}$

This will be called the domain of p

$(X = \mathrm{dom}(p))$

Define $\mathbb{P}$ as the set of all $p = (X, \mathcal{S})$

where:

1) $X \in [\omega_1]^{<\omega}$

2) $\mathcal{S} = \langle S^p_\alpha \rangle_{\alpha \in X}$ where

$$S^p_\alpha : X \cap \alpha \longrightarrow 2$$

Let $p, q \in \mathbb{P}$. Define $p \leq q$ if:

1) $X_q \subseteq X_p$

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2) If $\alpha \in X_q$, then $S_\alpha^q \subseteq S_\alpha^p$

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1) $X_q \subseteq X_p$

2) If $\alpha \in X_q$, then $S_\alpha^q \subseteq S_\alpha^p$

3) If $\alpha, \beta \in X_q$, $\xi \in X_p \setminus X_q$ with $\xi < \alpha, \beta$

$$S_\alpha^p(\xi) = S_\beta^p(\xi)$$

We will define $\{p_F \mid F \in \mathcal{F}\}$ such that:

1) $dom(p_F) = F$

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2) If $E \subseteq F$, then $p_F \leq p_E$.

3) If $E, F \in \mathcal{F}_k$, then:

a) p_F and p_E are compatible

b) If $\xi, \alpha \in F$ with $\xi < \alpha$ and $\varphi = \varphi_{FE}$

then:
$$S^{E}_{\varphi(\alpha)}(\varphi(\xi)) = S^{F}_{\alpha}(\xi)$$

We define the condition recursively

$\boxed{\text{base case: } k=0}$

Let $F \in \mathcal{F}_0$, say $F = \{\alpha\}$. Define:

$$P_F = C(\{\alpha\}, \{S_\alpha^F\})$$

where $S_\alpha^F = \phi$

Recursive step:

Assume we defined p_E for $E \in \mathcal{F}_k$.

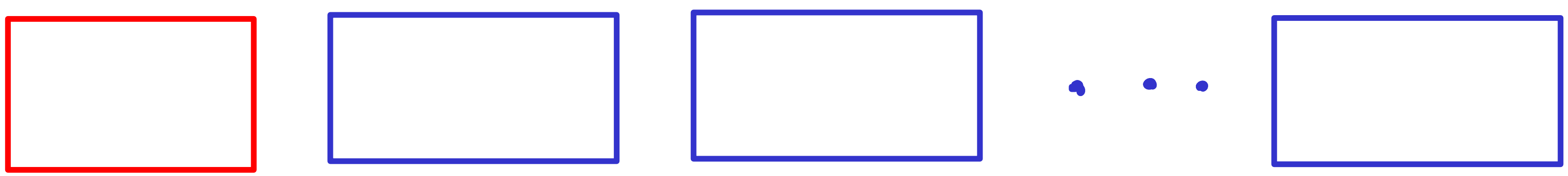
We will now define for the element in $\mathcal{F}_{k+1}$

We proceed by cases, depending in which part of the partition we are

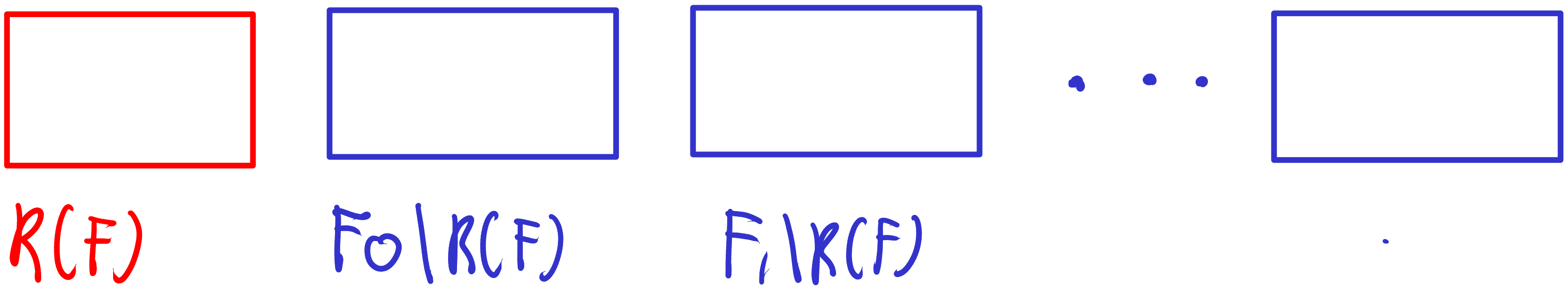
Case: $k+1 \in P_c$

We will kill uncountable chains

Let $F \in \mathcal{F}_{k+1}$ and $\{F_i \mid i < n_{k+1}\}$ the canonical decomposition.



Let $\alpha \in F_i$. We need to extend $S_\alpha^{F_i}$ to a function on $F \cap \alpha$



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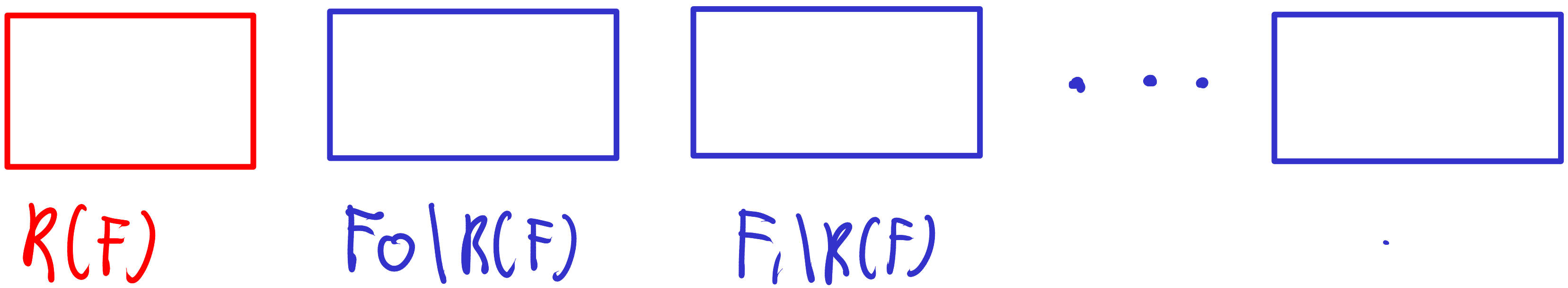
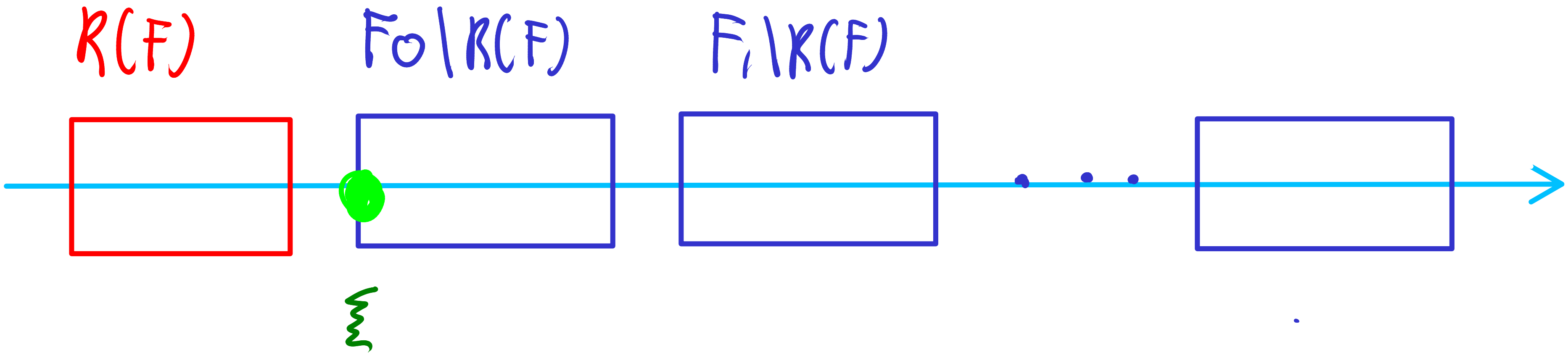


Note that if $\alpha \in F_0$, there is nothing to do

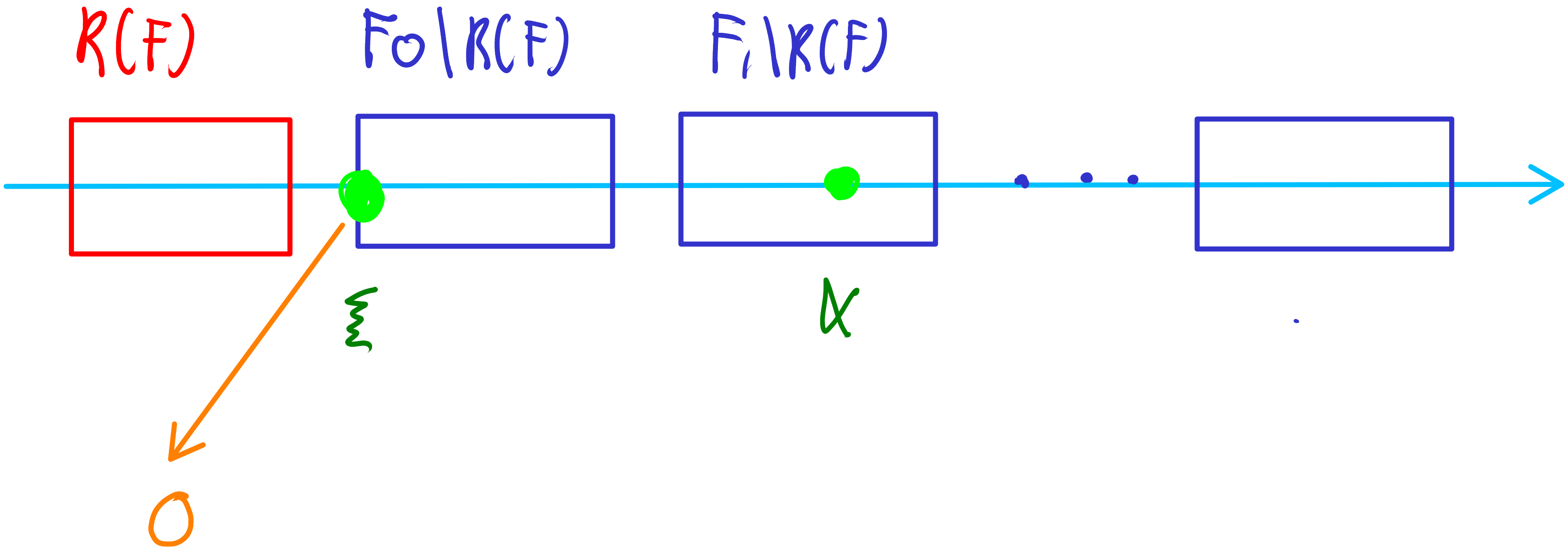
Let $\xi = \min(F_0 \setminus R(F))$



Let $\alpha \in F_i \setminus R(F)$ with $i > 0$. We do the following:

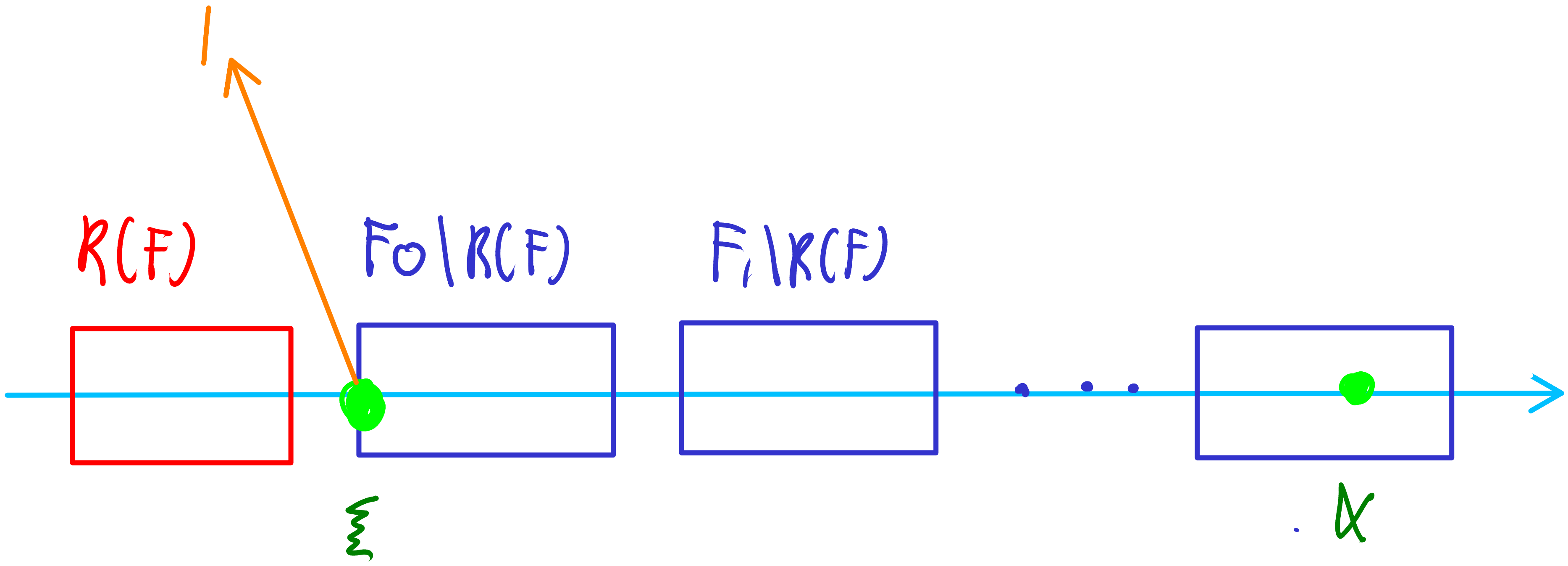
1) If $\alpha \in F_1 \setminus R(F)$, then define

$$S_\alpha^F(\xi) = 0$$



1) If $\alpha \in F_i \setminus R(F)$ with $i > 1$, define

$S_\alpha^F(\xi) = 1$



In all other place that need to be define, define it as 0.

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The point is that if $\alpha \in F_1 \setminus K(F)$ and $\beta \in F_2 \setminus K(F)$, then

$$S_\alpha^F(\xi) \neq S_\beta^F(\xi)$$

Case: $k+1 \in P_a$

We will kill uncountable antichains

Let $F \in \mathcal{F}_{k+1}$ and $\{F_i \mid i < n_{k+1}\}$ the canonical decomposition.

Recall that $n_{k+1} \geq 2^{(m_k - r_k)} + 1$

Let $\overline{F_0} = F_0 \setminus R(F)$, which has size

$2^{(m_k - r_k)}$

Enumerate the first $2^{(m_k - r_k)} + 1$ elements of the canonical decomposition as:

$$\{F_i \mid i < 2^{m_k - r_k} + 1\} = \{F_0\} \cup \{F_g \mid g : \overline{F_0} \to 2\}$$

Let $\alpha \in F_g$. Define S_α^F such that:

1) $S_\alpha^{F_g} \subseteq S_\alpha^F$

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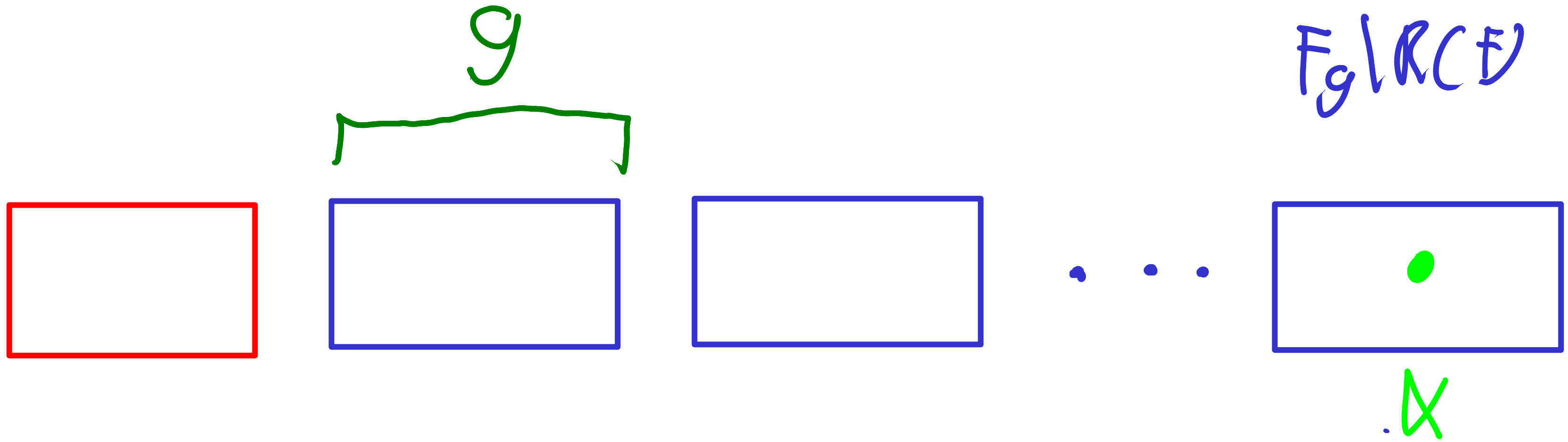


Let $\alpha \in F_g$. Define S_α^F such that:

1) $S_\alpha^{F_g} \subseteq S_\alpha^F$

2) $S_\alpha^F \restriction F_0 = g$

3) Make it 0 elsewhere if needed

If α is not in the first $2^{m_k - r_k}+1$ blocks, define it 0 everywhere where is needed

This finishes the recursive construction

We look at $\{p_F \mid F \in \mathcal{F}\}$

For $\alpha < \omega_1$, define:

$$e_\alpha = \bigcup \{ s_\alpha^F \mid \alpha \in F \in \mathcal{F} \}$$

Exercise

$\bar{e} = \langle e_\alpha \rangle_{\alpha < \omega_1}$ is a coherent sequence

We now need to prove that

$T(\bar{e})$ has no uncountable chains or antichains

recall

$T(\bar{e}) = \{ e_\alpha \restriction \xi \mid \xi < \alpha < \omega_1 \}$

$\boxed{T(\bar{e}) \text{ has no uncountable chain}}$

We proceed by contradiction. Assume that we have an uncountable chain:

$$C = \{ e_\alpha \upharpoonright \eta_\alpha \mid \alpha \in W \}$$

for some $W \in [\omega_1]^{\omega_1}$ and $\eta_\alpha \leq \alpha$

Since the level of $T(\bar{e})$ are countable (by shrinkng W if needed), we may assume that:

$$\alpha < \beta \Rightarrow \alpha < \eta_\beta$$

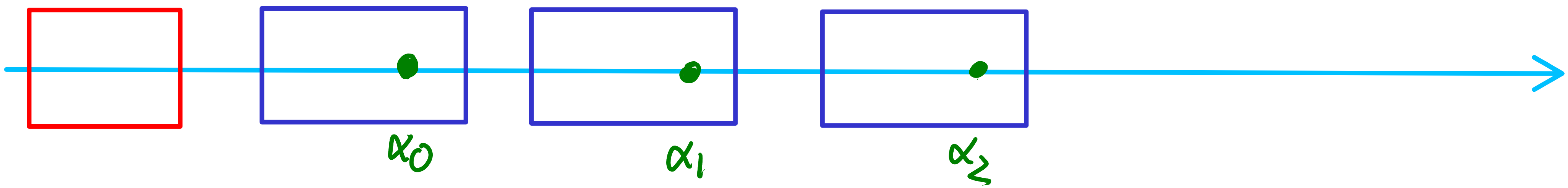
We now apply that $\mathcal{F}$ is

fully capturing

We capture W in P_c. In particular, we obtain:

$k+1 \in P_c$, $F \in \mathcal{F}_{k+1}$ and $\alpha_0 < \alpha_1 < \alpha_2$ in W such that:

F captures $\alpha_0, \alpha_1, \alpha_2$



x_0
x_1
x_2

Let $\xi = \min(F_0)$. By construction:

$$e_{\alpha_1}(\xi) \neq e_{\alpha_2}(\xi)$$

Moreover:

$$\xi \leq \aleph_0 < \eta_{\alpha_1}, \eta_\alpha$$

so $e_{\alpha_1} \upharpoonright \eta_{\alpha_1}$ is not an initial segment of $e_{\alpha_2} \upharpoonright \eta_{\alpha_2}$, which contradicts that C is a chain!

$\boxed{T(\bar{e}) \text{ has no uncountable antichain}}$

It is enough to prove the following:

$\boxed{T(\bar{e}) \text{ has no uncountable antichain}}$

It is enough to prove the following:

For every $W \in [\omega_1]^{\omega_1}$, there are $\alpha, \beta \in W$ with $\alpha < \beta$ such that $e_\alpha \subseteq e_\beta$

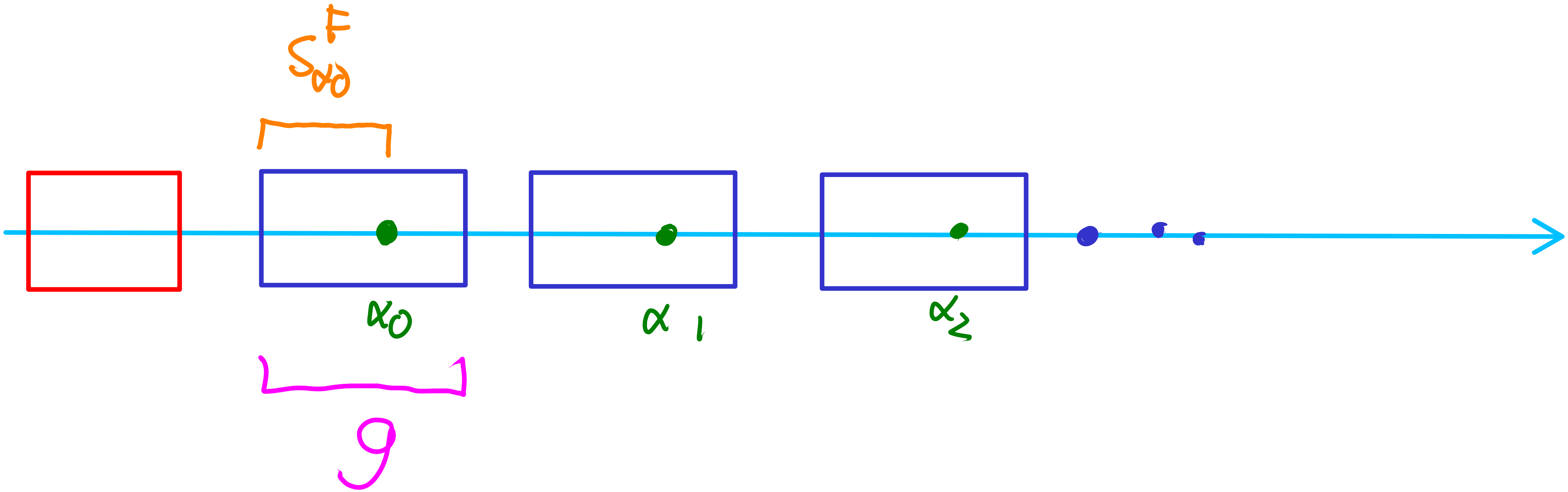
Now, using full capturing we can find

$k+1 \in P_a$, $F \in \mathcal{F}_{k+1}$ and $\{\alpha_i \mid i < n_{k+1}\} \subseteq W$

such that F fully captures it

Let $g : \bar{F}_0 \longrightarrow 2$ such that:

$$s^F_{\alpha_0} \restriction \bar{F}_0 \subseteq g$$



By construction, it follows that

$$e_{\alpha_0} \subseteq e_{\alpha_g}$$

$\square$

There are more applications

and much more to discover!

Thank you!